Operations on Power Series – Recall that $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$. Using this series, find the power series representation of the following expressions.

1. $\dfrac{x^5}{1-x}$

$$\frac{x^5}{1-x} = (x^5)\frac{1}{1-x} = x^5 \sum_{k=0}^{\infty} x^k = \sum_{k=0}^{\infty} x^5 x^k = \boxed{\sum_{k=0}^{\infty} x^{5+k}}$$

2. $\dfrac{1}{1+x^2}$

Let $u = -x^2$

$$\Rightarrow \frac{1}{1-u} = \sum_{k=0}^{\infty} u^k = \sum_{k=0}^{\infty} (-x^2)^k = \boxed{\sum_{k=0}^{\infty} (-1)^k x^{2k}}$$

3. $\dfrac{d}{dx}\dfrac{1}{1-x}$

$$\frac{d}{dx}\sum_{k=0}^{\infty} x^n = \frac{d}{dx}x^0 + \frac{d}{dx}x^1 + \frac{d}{dx}x^2 + \dots + \frac{d}{dx}x^n$$

$$= 0 + 1 + 2x + \dots nx^{n-1}$$

$$\Rightarrow \frac{d}{dx}\frac{1}{1-x} = \boxed{\sum_{k=0}^{\infty} kx^{n-1}}$$

Maclaurin Series – Find the Maclaurin series and interval of convergence for each of the following functions.

4. $f(x) = \cos x$

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k \Rightarrow \cos(x) \approx 1 + \frac{0}{1!}x^1 + \frac{-1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4 + \dots$$

$f'(x) = -\sin(x) = 0$

$f''(x) = -\cos(x) = -1$

$f^{(3)}(x) = \sin(x) = 0$

$f^{(4)}(x) = \cos(x) = 1$

...

$$\Rightarrow \cos(x) = \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}x^{2k}}$$

$$r = \lim_{k\to\infty}\left|\frac{(-1)^{k+1} x^{2(k+1)}}{(2(k+1))!} \cdot \frac{(2k)!}{(-1)^k x^{2k}}\right| = \lim_{k\to\infty}\left|\frac{x^2}{(2k+1)}\right| = 0 < 1 \Rightarrow \boxed{\text{Converges for all } x \text{ by ratio test}}$$

5. $g(x) = e^{2x}$

We know $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ for all real x

$$\Rightarrow e^{2x} = \boxed{\sum_{k=0}^{\infty} \frac{(2x)^k}{k!} \text{ for all real } x}$$

6. Evaluating a Limit by Taylor Series – Evaluate $\lim_{x\to\infty} 6x^5 \sin\frac{1}{x} - 6x^4 + x^2$

Let $t = \frac{1}{x}$ $\Rightarrow$ $\lim_{x\to\infty}\left[6x^5 \sin\left(\frac{1}{x}\right) - 6x^4 + x^2\right] = \lim_{t\to 0}\left[6t^{-5}\sin(t) - 6t^{-4} + t^{-2}\right] = \lim_{t\to 0}\left[\frac{6\sin(t) - 6t + t^3}{t^5}\right]$

$= \lim_{t\to 0}\left[\frac{6\left(t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040} + \dots\right) - 6t + t^3}{t^5}\right] = \lim_{t\to 0}\left[\frac{\left(\cancel{6t} - \cancel{\frac{6t^3}{6}} + \frac{6t^5}{120} - \frac{6t^7}{5040} + \dots\right) - \cancel{6t} + \cancel{t^3}}{t^5}\right]$

$= \lim_{t\to 0}\left[\frac{\frac{t^5}{20} - \frac{t^7}{840} + \dots}{t^5}\right] = \lim_{t\to 0}\left[\frac{1}{20} - \frac{t^2}{840} + \dots\right] = \boxed{\frac{1}{20}}$

REVIEW

Convergence of Series – Pick a test and determine if each of the following series converges. If it is an alternating series, determine if the convergence is absolute or conditional.

7. $\sum_{k=0}^{\infty} \frac{k}{2k+1}$

Use divergence test

$\lim_{k\to\infty} \frac{k}{2k+1} = \lim_{k\to\infty} \frac{\cancel{k}}{\cancel{k}(2+\frac{1}{k})} = \lim_{k\to\infty} \frac{1}{2+\frac{1}{k}} \quad \frac{1}{2+0} = \frac{1}{2} \neq 0 \Rightarrow$ This series diverges by the divergence test

8. $\sum_{k=1}^{\infty} \frac{k^2}{4^k}$

Use root test

$\lim_{k\to\infty} \sqrt[k]{\frac{k^2}{4^k}} = \lim_{k\to\infty} \frac{k^{\frac{2}{k}}}{4} = \frac{\lim_{k\to\infty}\left(k^{\frac{1}{k}}\right)^2}{4} = \frac{1}{4} < 1 \Rightarrow$ This series converges by root test

9. $\sum_{k=1}^{\infty} \frac{\cos k}{k^3}$

10. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{\frac{3}{2}}}$

Taylor Polynomials – Find the 2nd-order Taylor polynomial centered at 0 for the following functions.

11. $f(x) = \ln(x-1)$, center at $x=2$

$f(x) = \ln(x-1)$, $f(2) = \ln(2-1) = 0$

$f'(x) = \frac{1}{x-1}$, $f'(2) = \frac{1}{2-1} = 1$

$f''(x) = -\frac{1}{(x-1)^2}$, $f''(2) = -\frac{1}{(2-1)^2} = -1$

$$P_2(x) = f(2) + \frac{f'(2)}{1!}(x-2) + \frac{f''(2)}{2!}(x-2)^2$$

$$= 0 + \frac{1}{1!}(x-2) + \frac{-1}{2!}(x-2)^2$$

$$= \boxed{x-2-\tfrac{1}{2}(x-2)^2}$$

12. $g(x) = \tan x$

$g(x) = \tan(x)$, $g(0) = 0$

$g'(x) = \frac{1}{\cos^2(x)}$, $g'(0) = \frac{1}{\cos^2(0)} = 1$

$g''(x) = \frac{2\tan(x)}{\cos^2(x)}$, $g''(0) = \frac{2\tan(0)}{\cos^2(0)} = 0$

$$P_2(x) = \cancel{g(0)} + \frac{g'(0)}{1}x + \cancel{\frac{g''(0)}{2}x^2}$$

$$= \boxed{x}$$

Estimating Real Numbers – Estimate the value of the following numbers using a 2nd-order Taylor polynomial of your choice. Center the polynomial at the closest known value of the function you chose.

13. $(7.5)^{\frac{1}{3}}$ Choose $f(x) = x^{\frac{1}{3}}$, $c = 8$

$f(x) = x^{\frac{1}{3}}$, $f(8) = 2$

$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$, $f'(8) = \frac{1}{3}\frac{1}{\sqrt[3]{64}} = \frac{1}{12}$

$f''(x) = -\frac{2}{9}x^{-\frac{5}{3}}$, $f''(8) = -\frac{2}{9}\frac{1}{\sqrt[3]{8^5}} = -\frac{1}{144}$

$$P_2(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$$

$$\Rightarrow P_2(7.5) = 2 + \frac{1}{12}(7.5-8) - \frac{1}{288}(7.5-8)^2$$

$$= \boxed{1.957465277777}$$

14. $\sqrt{3.9}$ Choose $f(x) = x^{\frac{1}{2}}$, $c = 4$

$f(x) = x^{\frac{1}{2}}$, $f(4) = 2$

$f'(x) = -\frac{1}{2}x^{-\frac{1}{2}}$, $f'(4) = -\frac{1}{2}\frac{1}{(4)^{\frac{1}{2}}} = -\frac{1}{4}$

$f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}$, $f''(4) = -\frac{1}{4}\frac{1}{\sqrt{4^3}} = -\frac{1}{32}$

$$P_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$$

$$\Rightarrow P_2(3.9) = 2 + \frac{1}{4}(3.9-4) - \frac{1}{64}(3.9-4)^2$$

$$= \boxed{1.97484375000}$$