Improper Integrals – Evaluate the following definite integrals, or prove that they diverge

1. $\int_0^\infty e^{-3x}\,dx$ $= \lim_{b\to\infty} \int_0^b e^{-3x}\,dx = \lim_{b\to\infty}\left(-\frac{1}{3}e^{-3x}\right)\Big|_0^b = \lim_{b\to\infty} \frac{1}{3}\left(1-\frac{1}{e^{3b}}\right) = \frac{1}{3}\left(1-\frac{1}{\infty}\right)$

$= \boxed{\frac{1}{3}}$

2. $\int_1^\infty 1 + x^{-1}\,dx$

By p series theorem, $\int_1^\infty \frac{1}{x^p}dx$ converges for $p>1$, and diverges otherwise, for this $p=1$

$\therefore$ $\boxed{\text{diverges}}$

Hint: This integral is a standard form – check your textbook for a formula!

3. $\int_1^\infty \frac{1}{x^4}\,dx$

By p-series theorem, $\int_1^\infty \frac{1}{x^p}dx = \frac{1}{p-1}$ for $p>1$

$\Rightarrow \int_1^\infty \frac{1}{x^4}dx = \frac{1}{4-1} = \boxed{\frac{1}{3}}$

4. $\int_0^3 \frac{2}{\sqrt{9-x^2}}\,dx$ $= 2\lim_{c\to 3^-}\int_0^c \frac{1}{\sqrt{3^2-x^2}}dx = 2\lim_{c\to 3^-} \operatorname{asin}\left(\frac{x}{3}\right)\Big|_0^c$

$= 2\left(\operatorname{asin}\left(\frac{3}{3}\right) - \operatorname{asin}(0)\right)$

$= 2\left(\frac{\pi}{2} - 0\right) = \boxed{\pi}$

Differential Equations – Find a function, *f*, that satisfies each set of conditions

$f'(x) = 10e^{-\frac{x}{2}}, \qquad f(0) = 4$

$f(x) = \int f'(x)\,dx = \int 10e^{-\frac{x}{2}}\,dx = -20e^{-\frac{x}{2}} + C$

set $x = 0$, $f = 4$

$\Rightarrow 4 = -20e^{-\frac{0}{2}} + C \Rightarrow C = 24$

$\Rightarrow \boxed{f(x) = -20e^{-\frac{x}{2}} + 24}$

$\frac{df}{dx} = f^2 e^{-x}, \qquad f(0) = \frac{1}{2}$

$\int \frac{1}{f^2}\frac{df}{dx}\,dx = \int e^{-x}\,dx \Rightarrow \int \frac{1}{f^2}\,df = \int e^{-x}\,dx \Rightarrow -f^{-1} = -e^{-x} + C \Rightarrow f^{-1} = e^{-x} + C$

$\Rightarrow f(x) = \frac{1}{e^{-x} + C} \Rightarrow \frac{1}{2} = \frac{1}{e^{0} + C} \Rightarrow C = 1$

$\Rightarrow \boxed{f(x) = \frac{1}{e^{-x} + 1}}$