Absolute and conditional convergence – Determine whether the following series diverge, converge absolutely, or converge conditionally.

1. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$

Abs. convergence:

$\sum_{k=1}^{\infty} \frac{1}{k^{\frac{1}{2}}}$ diverges by p-series test

Cond. convergence:

$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{\frac{1}{2}}}$ Magnitude of $1/\sqrt{k}$ decreases? ✓ ; $\lim_{k\to\infty} 1/\sqrt{k} = 0$? ✓ $\Big\}$ converges by alternating series test

$\therefore \boxed{\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}} \text{ converges conditionally}}$

2. $\sum_{k=1}^{\infty} \frac{\sin k}{k^2}$

$\left|\frac{\sin(k)}{k^2}\right| = \frac{|\sin(k)|}{k^2} \leq \frac{1}{k^2}$

$\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges by p-series test $\therefore \sum_{k=1}^{\infty} \left|\frac{\sin(k)}{k^2}\right|$ converges by comparison test

$\therefore \boxed{\sum_{k=1}^{\infty} \frac{\sin(k)}{k^2} \text{ converges absolutely}}$

Taylor polynomials – Find the Taylor polynomials p_1 through p_7 centered at c for the following functions.

3. $f(x) = \sin(x), c = 0$

$f(x) = \sin(x)$, $f(0) = \sin(0) = 0$

$f'(x) = \cos(x)$, $f'(0) = \cos(0) = 1$

$f''(x) = -\sin(x)$, $f''(0) = -\sin(0) = 0$

$f^{(3)}(x) = -\cos(x)$, $f^{(3)}(0) = -\cos(0) = -1$

$f^{(4)}(x) = \sin(x)$, $f^{(4)}(0) = \sin(0) = 0$

$f^{(5)}(x) = \cos(x)$, $f^{(5)}(0) = \cos(0) = 1$

$f^{(6)}(x) = -\sin(x)$, $f^{(6)}(0) = -\sin(0) = 0$

$f^{(7)}(x) = -\cos(x)$, $f^{(7)}(0) = -\cos(0) = -1$

$$P_7(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f^{(3)}(0)}{3!}(x-0)^3 + \frac{f^{(4)}(0)}{4!}(x-0)^4 + \frac{f^{(5)}(0)}{5!}(x-0)^5 + \frac{f^{(6)}(0)}{6!}(x-0)^6 + \frac{f^{(7)}(0)}{7!}(x-0)^7$$

$$= \boxed{0 + x + 0 - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7}$$

Truncate appropriately for $P_1 - P_6$

4. $g(x) = \ln x, c = 1$

$g(x) = \ln(x)$, $g(1) = \ln(1) = 0$

$g'(x) = \frac{1}{x}$, $g'(1) = \frac{1}{1} = 1$

$g''(x) = -\frac{1}{x^2}$, $g''(1) = -\frac{1}{1^2} = -1$

$g^{(3)}(x) = \frac{2}{x^3}$, $g^{(3)}(1) = \frac{2}{1^3} = 2$

$g^{(4)}(x) = -\frac{6}{x^4}$, $g^{(4)}(1) = -\frac{6}{1^4} = -6$

$g^{(5)}(x) = \frac{24}{x^5}$, $g^{(5)}(1) = \frac{24}{1^5} = 24$

$g^{(6)}(x) = -\frac{120}{x^6}$, $g^{(6)}(1) = -\frac{120}{1^6} = -120$

$g^{(7)}(x) = \frac{720}{x^7}$, $g^{(7)}(1) = \frac{720}{1^7} = 720$

$$P_7(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f^{(3)}(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4 + \frac{f^{(5)}(1)}{5!}(x-1)^5 + \frac{f^{(6)}(1)}{6!}(x-1)^6 + \frac{f^{(7)}(1)}{7!}(x-1)^7$$

$$= 0 + (x-1) - \frac{1}{2}(x-1)^2 + \frac{2}{6}(x-1)^3 + \frac{6}{24}(x-1)^4 + \frac{24}{120}(x-1)^5 + \frac{120}{720}(x-1)^6 + \frac{720}{5040}(x-1)^7$$

$$= \boxed{x - 1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 - \frac{1}{6}(x-1)^6 + \frac{1}{7}(x-1)^7}$$

Again, truncate appropriately for $P_1 - P_6$

5. Estimating the remainder – Find a bound for the magnitude of the remainder for the Taylor polynomials of $f(x) = \cos x$ centered at 0.

By theorum 9.1: $|R_n(x)| = M \frac{|x-a|^{n+1}}{(n+1)!}$ $a=0$

$\therefore \quad R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$ $f^{(n+1)}(c) = \pm \cos(c), \pm \sin(c) \leq 1$

$\therefore \quad M = 1$

$$\boxed{|R_n(x)| \leq M \frac{|x|^{n+1}}{(n+1)!} = \frac{|x|^{n+1}}{(n+1)!}}$$